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LATTICES OF TWO-SIDED IDEALS OF LOCALLY MATRICIAL ALGEBRAS AND THE Γ-INVARIANT PROBLEM

BY

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To the memory of Igor Slatkovský

ABSTRACT

We develop a method of representation of distributive $(\vee, 0, 1)$-semilattices as semilattices of finitely generated ideals of locally matricial algebras. We use the method to reprove two representation results by G. M. Bergman and prove a new one that every distributive $(0, 1)$-lattice is, as a semilattice, isomorphic to the semilattice of all finitely generated ideals of a locally matricial algebra. We apply this fact to solve the Γ-invariant problem.

Introduction

A lattice is strongly dense provided it possesses a cofinal continuous strictly decreasing chain (abbreviated to c.d.c.) in the poset of its nonzero elements. The dimension of a strongly dense lattice is the length of its shortest c.d.c. If a modular strongly dense lattice L has dimension $\aleph_0$ then L possesses either a c.d.c. $(a_m \mid n < \omega)$ such that a_n is complemented over a_m for every $n < m$ (we say that L is complementing) or a c.d.c. $(a_m \mid n < \omega)$ such that a_n is not complemented over a_m for every $n < m$ (then we say that the lattice L is narrow). For strongly dense lattices of uncountable dimension κ is defined an invariant, called the Γ-invariant, which is an element of $\mathcal{B}(\kappa)$, the Boolean

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algebra of all subsets of κ modulo the filter generated by closed unbounded subsets. This invariant in some sense measures the failure of the lattice to be relatively complemented [ET].

Let $\overline{E}$ denote the element of $\mathcal{B}(\kappa)$ represented by a subset E of an uncountable regular cardinal κ. By [ET, Theorem 1.3], there exists a distributive strongly dense lattice of dimension (and cardinality) κ whose Γ-invariant is $\overline{E}$. Furthermore, the lattice I_E of all nonzero ideals of L_E is an algebraic distributive strongly dense lattice of dimension κ with the Γ-invariant $\overline{E}$.

A right module over an associative ring is strongly uniform provided its submodule lattice is strongly dense. The dimension and the Γ-invariant of a strongly uniform module are defined as the dimension and the Γ-invariant of its submodule lattice. J. Trlifaj [T1] studied possible values of the dimensions and the Γ-invariants of strongly uniform modules over rings of various types. In particular, he proved that every strongly uniform module over a commutative Noetherian ring is of finite or countable dimension and that in the latter case it is narrow [T1, Theorem 2.8]. Over commutative rings [T1, Theorem 2.10] or (noncommutative) Noetherian rings [T1, Example 2.11] there are strongly uniform modules of any uncountable dimension κ, but their only possible Γ-invariant is $\overline{\kappa}$. Finally, for every regular cardinal number κ, he found an example of a module of dimension κ over a unit-regular ring. The Γ-invariants of these modules were again $\overline{\kappa}$ and he asked about all the possible values of the Γ-invariants of strongly uniform modules over non-right perfect rings, in particular, over rings which are von Neumann regular [T1, Open problem 3]. This question will be referred to as the Γ-invariant problem.

Later on, P. C. Eklof and J. Trlifaj constructed a strongly dense module of a countable dimension which is complementing and more complex examples of strongly uniform modules of an uncountable dimension over a locally semisimple algebra (which is a unit-regular ring) [ET, Theorem 2.7] but the Γ-invariant problem remained open [ET, Problem 2.3].

The Γ-invariant problem was our original motivation. We have tried to apply the following idea [ET]: A ring R is a right module over the ring $R \otimes_{\mathbb{Z}} R^{op}$ (with the multiplication given by $t(r \otimes s) = str$) and submodules of this module correspond to two-sided ideals of the ring R. In general, regularity is not preserved by this tensor product construction but if R is a locally matricial algebra, then the ring $R \otimes_{\mathbb{Z}} R^{op}$ is a locally matricial algebra as well. Thus we focused on representations of algebraic lattices as the lattices of two-sided ideals of locally matricial algebras.

It is well known that the lattice of two-sided ideals of a von Neumann regular ring is distributive. G. M. Bergman [Be] proved that every algebraic distributive lattice either isomorphic to the lattice of lower subsets of a partially ordered set or with at most countably many compact elements is isomorphic to the two-sided ideal lattice of a locally matricial algebra. In contrast, F. Wehrung [W1, W2] constructed an algebraic distributive lattice with $\aleph_2$ compact elements which cannot be realized as the lattice of two-sided ideals of any von Neumann regular ring. Further, he proved that if an algebraic distributive lattice has $\aleph_1$ compact elements, then it can be realized as the lattice of two-sided ideals of a von Neumann regular rings [W3]; however, he proved recently that the result fails for locally matricial algebras [W4].

The main result of the paper is the realization of every algebraic distributive lattice whose compact elements form a lattice as the lattice of two-sided ideals of a locally matricial algebra [GW, Problem 1]. In particular, the lattice I_E has such a realization for every subset E of a regular cardinal κ, which leads to the solution of the Γ-invariant problem.

At the same time as we achieved this result, S. Shelah and J. Trlifaj [ST] constructed, for every regular cardinal κ and every subset E of κ, a vector space V over a given field k and a k-subalgebra R of the endomorphism ring of V such that V, as an R-module, is strongly uniform of dimension κ and its Γ-invariant equals $\overline{E}$. However, the ring R is not von Neumann regular.

Now, let us outline the organization of the paper. In the first two sections we develop tools for realization of distributive $(\vee, 0, 1)$-semilattices as semilattices of finitely generated ideals of unital locally matricial algebras. In Section 3 we use these tools to reprove Bergman's results. Section 4 is devoted to the proof of the main result and Section 5 to its application to the solution of the Γ-invariant problem.

Notation

The set of all natural numbers is denoted by ω. This notation is used also for the first infinite ordinal. Given a set M, we denote by $\mathcal{P}(M)$ the set of all subsets of M, and by $[M]^{<\omega}$ the set of all finite subsets of the set M. For a map $\varphi\colon M \to N$, we define a map $\mathcal{P}(\varphi)\colon \mathcal{P}(N) \to \mathcal{P}(M)$ by the correspondence $N' \mapsto \varphi^{-1}(N')$, where N' is a subset of N.

Let a be an element of a partially ordered set P. We use the notation

$$[a)_P = \{b \in P \mid a \leq b\}, \quad (a]_P = \{b \in P \mid b \leq a\}$$

for the lower, upper subset of P generated by the element a, respectively. We drop the subscript if the set P is understood.

Let $\mathbf{C}$ be a category. We denote by $\mathbf{C}(a, b)$ the set of all morphisms with domain a and codomain b. By $\mathbf{1}_a$, we denote the identity morphism of an object $a \in \mathbf{C}$. For all categories except the category $\mathbf{c}$ defined in Section 2, identity morphisms correspond to identity maps.

Let k be a field. Recall that a family $(V_i \mid i \in I)$ of subspaces of a k-vector space V is independent if for every $i \in I$, the intersection of V_i with the subspace of V spanned by $(V_j \mid j \in I \setminus \{i\})$ is the zero subspace. Given an independent family $(V_i \mid i \in I)$ of subspaces of a k-vector space V, we denote by $\bigoplus_{i \in i} V_i$ the subspace of V spanned by all the V_i, $i \in I$. Moreover, given a family $(f_i\colon V_i \to W \mid i \in I)$ of k-linear maps, we denote by $\bigoplus_{i \in I} f_i$ the unique k-linear map f from $\bigoplus_{i \in i} V_i$ to W such that $f \restriction V_i = f_i$ for every $i \in I$.

1. Distributive semilattices

Lattices of substructures, congruences, ideals, etc. of algebraic structures are algebraic lattices [Gr, II.3. Definition 12]:

(i) Let L be a complete lattice and let a be an element of L. Then a is called **compact**, if $a \leq \bigvee X$, for some $X \subseteq L$, implies that $a \leq X_1$, for some finite $X_1 \subseteq X$.

(ii) A complete lattice is called **algebraic**, if every element is the join of compact elements.

The set of compact elements of a complete lattice L is closed under finite joins (not under finite meets in general) and contains the zero of L. Thus it forms a $(\vee, 0)$-semilattice, which we denote by L^{c}.

The ideal lattice of every $(\vee, 0)$-semilattice is algebraic. On the other hand, every algebraic lattice L is isomorphic to $\mathrm{Id}(L^{\mathrm{c}})$, the lattice of all nonempty ideals of the $(\vee, 0)$-semilattice L^{c} [Gr, II.3. Theorem 13].

A semilattice S is called **distributive** if $a \leq b_0 \vee b_1$ $(a, b_0, b_1 \in S)$ implies the existence of a_0, $a_1 \in S$ with $a_0 \leq b_0$, $a_1 \leq b_1$ and $a = a_0 \vee a_1$ [Gr, page 131]. A $(\vee, 0)$-semilattice S is distributive iff $\mathrm{Id}(S)$ (as a lattice) is distributive [Gr, II.5. Lemma 1, (iii)].

A nonzero element a of a distributive semilattice (resp. lattice) L is **join-irreducible**, if $a = b \vee c$ implies that either $a = b$ or $a = c$ for every b, $c \in L$. We denote by $J(L)$ the set of all join-irreducible elements of L, regarded as a partially ordered set under the partial ordering of L [Gr, page 81]. A subset H of a partially ordered set P is **hereditary**, if for every $b \in H$ and every $a \in P$,

$a \leq b$ implies that $a \in H$. We denote by $H(P)$ the set of all hereditary subsets of P. Observe that $H(P)$ with intersection and union as meet and join forms a distributive lattice. Every finite distributive semilattice (resp. lattice) L is isomorphic to the semilattice (resp. lattice) $H(J(L))$ of all hereditary subsets of $J(L)$ partially ordered by set inclusion [Gr, II.1. Theorem 9].

A finite distributive $(\vee, 0, 1)$-semilattice s is **Boolean**, if the order on the set $J(s)$ is trivial, that is, if s is isomorphic to the semilattice of all subsets of a finite set.

We denote by

- $\mathbf{s}$ — the category of all finite distributive $(\vee, 0, 1)$-semilattices (with $(\vee, 0, 1)$-preserving homomorphisms),
- $\mathbf{b}$ — the category of all finite Boolean semilattices (with $(\vee, 0, 1)$-preserving homomorphisms).

Given a finite distributive $(\vee, 0, 1)$-semilattice s, we denote by $Bo(s)$ the Boolean semilattice of all subsets of the set $J(s)$ and, for each $f \in \mathbf{s}(s_1, s_2)$, we define a homomorphism $Bo(f) \in \mathbf{b}(Bo(s_1), Bo(s_2))$ by the rule

$$Bo(f)(X) = \{j \in J(s_2) \mid j \leq f(\bigvee X)\} \quad (X \in Bo(s)).$$

Observe that Bo preserves the composition of morphisms but not the identity morphisms; indeed, $Bo(\mathbf{1}_s) = \mathbf{1}_{Bo(s)}$ iff s is Boolean.

Let s be a finite distributive $(\vee, 0, 1)$-semilattice. We define a pair of semilattice homomorphisms $K_s \colon s \to Bo(s)$ and $L_s \colon Bo(s) \to s$ by

$$K_s(x) = \{j \in J(s) \mid j \leq x\} \quad (x \in s)$$

and

$$L_s(X) = \bigvee X \quad (X \in Bo(s)).$$

Observe that

$$L_s \circ K_s = \mathbf{1}_s \tag{1.1}$$

and that for every homomorphism $f \in \mathbf{s}(s_1, s_2)$ the equalities

$$Bo(f) \circ K_{s_1} = K_{s_2} \circ f, \tag{1.2}$$

$$f \circ L_{s_1} = L_{s_2} \circ Bo(f) \tag{1.3}$$

and

$$K_{s_2} \circ f \circ L_{s_1} = Bo(f) \tag{1.4}$$

hold.

PROPOSITION 1.1: *Let P be an upwards directed partially ordered set without maximal elements and let*

$$\langle s_p, f_{p,q}\rangle_{p\leq q \text{ in } P}$$

be a direct system in s. *If*

$$\langle S, f_p\rangle_{p\in P} = \varinjlim \langle s_p, f_{p,q}\rangle_{p\leq q \text{ in } P},$$

then

$$\langle S, f_p \circ L_{s_p}\rangle_{p\in P} = \varinjlim \langle Bo(s_p), Bo(f_{p,q})\rangle_{p<q \text{ in } P}.$$

Proof: For all $p \in P$, put $L_p = L_{s_p}$, $K_p = K_{s_p}$, and $g_p = f_p \circ L_{s_p}$. For each pair $p < q$ in P, set $g_{p,q} = Bo(f_{p,q})$.

For all $p < q$ in P,

$$g_p = f_p \circ L_p = f_q \circ f_{p,q} \circ L_p = f_q \circ L_q \circ g_{p,q} = g_q \circ g_{p,q},$$

by (1.3). Let $\langle T, g'_p\rangle_{p\in P}$ be such that for every $p < q$ in P,

$$g'_p = g'_q \circ g_{p,q}.$$

We show that there exists exactly one $(\vee, 0, 1)$-semilattice homomorphism $h: S \to T$ such that $h \circ g_p = g'_p$ for every $p \in P$.

Put $f'_p = g'_p \circ K_p$ for all $p \in P$. Then

$$f'_q \circ f_{p,q} = g'_q \circ K_q \circ f_{p,q} = g'_q \circ g_{p,q} \circ K_p = g'_p \circ K_p = f'_p$$

for every $p < q$ in P by (1.2). Then, since $\langle S, f_p\rangle_{p\in P}$ is a direct limit of the direct system $\langle s_p, f_{p,q}\rangle_{p\leq q \text{ in } P}$, there exists a unique homomorphism h: $S \to T$ such that

$$h \circ f_p = f'_p$$

for every $p \in P$. It follows that for every $p < q$ in P,

$$\begin{aligned} h \circ g_p &= h \circ f_p \circ L_p = f'_p \circ L_p = g'_p \circ K_p \circ L_p = g'_q \circ g_{p,q} \circ K_p \circ L_p \\ &= g'_q \circ K_q \circ f_{p,q} \circ L_p = g'_q \circ g_{p,q} = g'_p \end{aligned}$$

(the 5^{th} equality is due to (1.2), the 6^{th} equality is due to (1.4)). Suppose that h': $S \to T$ is a $(\vee, 0, 1)$-semilattice homomorphism satisfying $h' \circ g_p = g'_p$ for every $p \in P$. Then

$$h' \circ g_p \circ K_p = g'_p \circ K_p \quad (p \in P),$$

hence

$$h' \circ f_p \circ L_p \circ K_p = f'_p \quad (p \in P),$$

and so, by (1.1),

$$h' \circ f_p = f'_p,$$

for every $p \in P$. It follows that $h = h'$. ∎

P. Pudlák [Pu] proved that every distributive $(\vee, 0)$-semilattice is the directed union of all its finite distributive $(\vee, 0)$-subsemilattices. Consequently, every distributive $(\vee, 0, 1)$-semilattice is a direct limit of a direct system $\mathcal{S}$ of finite distributive semilattices and $(\vee, 0, 1)$-preserving embeddings. Furthermore, we can assume that $\mathcal{S}$ is indexed by an upwards directed partially ordered set without maximal elements. Then, as a corollary of Proposition 1.1, we obtain the following result of K. R. Goodearl and F. Wehrung [GW, Theorem 6.6].

COROLLARY 1.2: *Every distributive $(\vee, 0, 1)$-semilattice is a direct limit of Boolean semilattices (and $(\vee, 0, 1)$-preserving homomorphisms).*

2. The category c

All rings are associative with a unit element; all ring homomorphisms are supposed to preserve the unit. For a ring R, we denote by $\mathrm{Id}(R)$ the lattice of two-sided ideals of R and by $\mathrm{Id}^{c}(R)$ the semilattice of compact elements of the lattice $\mathrm{Id}(R)$, that is, the semilattice of finitely generated two-sided ideals of R. Notice that $\mathrm{Id}^{c}(R)$ is a $(\vee, 0, 1)$-semilattice.

Given a ring homomorphism $\varphi\colon R \to S$, we define a map $\mathrm{Id}^{c}(\varphi)\colon \mathrm{Id}^{c}(R) \to \mathrm{Id}^{c}(S)$ by the correspondence

$$I \mapsto S\varphi(I)S. \tag{2.1}$$

The map $\mathrm{Id}^{c}(\varphi)$ is a $(\vee, 0, 1)$-semilattice homomorphism, and it is straightforward to verify that Id^{c} is a direct limit preserving functor from the category of rings to the category of $(\vee, 0, 1)$-semilattices.

The following example shows that it is not possible to define, in a similar way, a functor Id from the category of rings to the category of all algebraic lattices.

Example 2.1: Let k be a field, let $R = k \times k$ and $S = k \times \mathbb{M}_2(k)$ be k-algebras. Put $e_1 = (1, 0)$, $e_2 = (0, 1)$, and

$$f = \left(1, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right), \quad g_1 = \left(0, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right), \quad g_2 = \left(0, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right).$$

Denote by I_1, I_2 the two-sided ideals of R generated by primitive idempotents e_1, e_2, respectively, and by J the two-sided ideal of S generated by g_2. Let

$\varphi\colon R \to S$ be the ring homomorphism defined on the generators e_1, e_2 of R by $\varphi(e_1) = f + g_1$, $\varphi(e_2) = g_2$. Then correspondence (2.1) assigns to the ideal I_1 the whole ring S and the ideal I_2 is mapped to J. Since $I_1 \cap I_2 = 0$, while $S \cap J = J$, the map Id^{c} does not preserve finite meets.

Let k be a field. A **matricial** k-algebra R is a k-algebra of the form

$$\mathbb{M}_{p(1)}(k) \times \cdots \times \mathbb{M}_{p(n)}(k)$$

for some natural numbers $p(1), \dots, p(n)$ [Go, page 217]. The semilattice $\mathrm{Id}^{\mathrm{c}}(R)$ of all finitely generated two-sided ideals of the matricial algebra R is isomorphic to the Boolean semilattice of all subsets of the set $\{1, \dots, n\}$. We fix a field k and denote by $\mathbf{m}$ the category of all matricial k-algebras. Recall that a k-algebra is **locally matricial** provided it is a direct limit of matricial k-algebras.

In this section we shall define a new category $\mathbf{c}$ and a pair of functors $A\colon \mathbf{c} \to \mathbf{m}$ and $\Lambda\colon \mathbf{c} \to \mathbf{s}$ such that there is a natural isomorphism $\eta\colon \mathrm{Id}^{\mathrm{c}} A \to \Lambda$.

Definition: An **object** B of the category $\mathbf{c}$ consists of a finite set I and a family $(B^i \mid i \in I)$ of nonempty pairwise disjoint finite sets.

Let $B_1 = (B_1^i \mid i \in I_1)$, $B_2 = (B_2^j \mid j \in I_2)$ be objects of the category $\mathbf{c}$. A **premorphism** $B_1 \to B_2$ is a pair (C, h), where $C = (C^{i,j} \mid i \in I_1,\ j \in I_2)$ is a family of (possibly empty) finite sets and $h = (h^j \mid j \in I_2)$ is a family of bijections

$$h^j\colon \bigcup_{i \in I_1} (C^{i,j} \times B_1^i) \xrightarrow{\simeq} B_2^j.$$

We denote by $\mathbf{c}'(B_1, B_2)$ the collection of all premorphisms $B_1 \to B_2$.

We say that premorphisms (C, h), $(\widetilde{C}, \widetilde{h}) \in \mathbf{c}'(B_1, B_2)$ are equivalent (we write $(C, h) \sim (\widetilde{C}, \widetilde{h})$) if there is a collection $(g^{i,j}\colon C^{i,j} \to \widetilde{C}^{i,j} \mid i \in I_1, j \in I_2)$ of maps such that for every $i \in I_1$, $j \in I_2$, and for every $c \in C^{i,j}$, $b \in B^i$,

$$h^j(c, b) = \widetilde{h}^j(g^{i,j}(c), b). \tag{2.2}$$

Observe that the maps $g^{i,j}$, $i \in I_1$, $j \in I_2$ satisfying (2.2) are necessarily bijections. The morphisms in $\mathbf{c}$ are the equivalence classes with respect to the equivalence relation $\sim$, that is

$$\mathbf{c}(B_1, B_2) = \mathbf{c}'(B_1, B_2)/\sim.$$

We denote by $[C, h]$, or sometimes $[(C, h)]$, the equivalence class represented by the premorphism (C, h). We say that $[C, h]$ is a morphism from B_1 to B_2.

Now we shall define the composition of morphisms in $\mathbf{c}$. First we describe how the premorphisms are composed. For objects $B_1 = (B_1^i \mid i \in I_1)$, $B_2 = (B_2^j \mid j \in I_2)$, $B_3 = (B_3^k \mid k \in I_3)$ of the category $\mathbf{c}$ and premorphisms $(C_1, h_1) \in \mathbf{c}'(B_1, B_2)$, $(C_2, h_2) \in \mathbf{c}'(B_2, B_3)$, the composition $(C, h) = (C_2, h_2) \circ (C_1, h_1)$ consists of the family $C = (C^{i,k} \mid i \in I_1, k \in I_3)$ of sets, resp. a family $h = (h^k \mid k \in I_3)$ of maps defined by

$$C^{i,k} = \bigcup_{j \in I_2} (C_2^{j,k} \times C_1^{i,j})$$

for every $i \in I_1$, $k \in I_3$, resp.

$$h^k((c_2, c_1), b) = h_2^k(c_2, h_1^j(c_1, b))$$

for every $b \in B_1^i$, $c_1 \in C_1^{i,j}$, $c_2 \in C_2^{j,k}$, where $i \in I_1$, $j \in I_2$, and $k \in I_3$.

LEMMA 2.2: *Let $B_1 = (B_1^i \mid i \in I_1)$, $B_2 = (B_2^j \mid j \in I_2)$, and $B_3 = (B_3^k \mid k \in I_3)$ be objects of the category $\mathbf{c}$. Let (C_1, h_1), $(\widetilde{C}_1, \widetilde{h}_1) \in \mathbf{c}'(B_1, B_2)$ and (C_2, h_2), $(\widetilde{C}_2, \widetilde{h}_2) \in \mathbf{c}'(B_2, B_3)$. If $(C_1, h_1) \sim (\widetilde{C}_1, \widetilde{h}_1)$ and $(C_2, h_2) \sim (\widetilde{C}_2, \widetilde{h}_2)$, then*

$$(C_2, h_2) \circ (C_1, h_1) \sim (\widetilde{C}_2, \widetilde{h}_2) \circ (\widetilde{C}_1, \widetilde{h}_1).$$

Proof: Since $(C_1, h_1) \sim (\widetilde{C}_1, \widetilde{h}_1)$, there are maps

$$g_1^{i,j} \colon C_1^{i,j} \to \widetilde{C}_1^{i,j} \quad (i \in I_1,\ j \in I_2)$$

such that for every $b \in B_1^i$ and $c \in C_1^{i,j}$,

$$h_1^j(c, b) = \widetilde{h}_1^j(g_1^{i,j}(c), b).$$

Similarly, since $(C_2, h_2) \sim (\widetilde{C}_2, \widetilde{h}_2)$, there are maps

$$g_2^{j,k} \colon C_2^{j,k} \to \widetilde{C}_2^{j,k} \quad (j \in I_2,\ k \in I_3)$$

such that for every $b \in B_2^j$ and $c \in C_2^{j,k}$,

$$h_2^k(c, b) = \widetilde{h}_2^k(g_2^{j,k}(c), b).$$

We put

$$g^{i,k} = \bigcup_{j \in I_2} (g_2^{j,k} \times g_1^{i,j}) \quad (i \in I_1,\ k \in I_3),$$

and we denote by (C, h), resp. $(\widetilde{C}, \widetilde{h})$ the composition $(C_2, h_2) \circ (C_1, h_1)$, resp. $(\widetilde{C}_2, \widetilde{h}_2) \circ (\widetilde{C}_1, \widetilde{h}_1)$. Then for every $b \in B_1^i$, $c_1 \in C_1^{i,j}$, and $c_2 \in C_2^{j,k}$, where $i \in I_1$, $j \in I_2$, and $k \in I_3$,

$$\begin{aligned} h^k((c_2, c_1), b) &= h_2^k(c_2, h_1^j(c_1, b)) = \widetilde{h}_2^k(g_2^{j,k}(c_2), \widetilde{h}_1^j(g_1^{i,j}(c_1), b)) \\ &= \widetilde{h}^k((g_2^{j,k}(c_2), g_1^{i,j}(c_1)), b) = \widetilde{h}^k(g^{i,k}(c_2, c_1), b). \end{aligned}$$ ∎

Let (C_2,h_2), (C_1,h_1) be premorphisms as above. Lemma 2.2 enables us to define

$$[(C_2,h_2)\circ(C_1,h_1)] = [(C_2,h_2)]\circ[(C_1,h_1)].$$

It remains to prove that the composition is associative and that every object of $\mathbf{c}$ possesses an identity morphism.

LEMMA 2.3: *The composition of morphisms is associative, that is, let $B_n = (B_n^i \mid i \in I_n)$, $n = 1,\dots,4$, be objects of the category $\mathbf{c}$ and let $[C_n,h_n] \in \mathbf{c}(B_n,B_{n+1})$ for $n = 1,2,3$. Then*

$$[C_3,h_3]\circ([C_2,h_2]\circ[C_1,h_1]) = ([C_3,h_3]\circ[C_2,h_2])\circ[C_1,h_1].$$

Proof: Put

$$(C,h) = (C_3,h_3)\circ((C_2,h_2)\circ(C_1,h_1))$$

and

$$(\widetilde{C},\widetilde{h}) = ((C_3,h_3)\circ(C_2,h_2))\circ(C_1,h_1).$$

We prove that

$$(C,h)\sim(\widetilde{C},\widetilde{h}). \tag{2.3}$$

It follows from the definition that for every $i \in I_1$, and $l \in I_4$,

$$C^{i,l} = \bigcup_{k\in I_3}\left(C_3^{k,l}\times\left(\bigcup_{j\in I_2}(C_2^{j,k}\times C_1^{i,j})\right)\right) = \bigcup_{j\in I_2}\bigcup_{k\in I_3}(C_3^{k,l}\times(C_2^{j,k}\times C_1^{i,j})),$$

while

$$\widetilde{C}^{i,l} = \bigcup_{j\in I_2}\left(\left(\bigcup_{k\in I_3}(C_3^{k,l}\times C_2^{j,k})\right)\times C_1^{i,j}\right) = \bigcup_{j\in I_2}\bigcup_{k\in I_3}((C_3^{k,l}\times C_2^{j,k})\times C_1^{i,j}).$$

It is straightforward to verify that for every $b \in B_1^i$, $c_1 \in C_1^{i,j}$, $c_2 \in C_2^{j,k}$, and $c_3 \in C_3^{k,l}$, where $i \in I_1$, $j \in I_2$, $k \in I_3$, and $l \in I_4$, the equality

$$h^l((c_3,(c_2,c_1)),b) = h_3^l(c_3,h_2^k(c_2,h_1^j(c_1,b))) = \widetilde{h}^l(((c_3,c_2),c_1),b) \tag{2.4}$$

holds. Finally, for all $i \in I_1$ and $l \in I_4$, define a bijection $g^{i,l}\colon C^{i,l}\to\widetilde{C}^{i,l}$ by the correspondence $(c_3,(c_2,c_1)) \mapsto ((c_3,c_2),c_1)$. Then, due to (2.4), for every $b \in B_1^i$ and every $c \in C^{i,l}$,

$$h^l(c,b) = \widetilde{h}^l(g^{i,l}(c),b).$$

This proves (2.3). ∎

Given an object $B = (B^i \mid i \in I)$ in the category $\mathbf{c}$, we put

$$C^{i,j} = \begin{cases} \emptyset, & \text{if } i \neq j \\ \{i\}, & \text{if } i = j \end{cases}$$

for every $i, j \in I$ and we define maps h^j, $j \in I$, from $\bigcup_{i \in I}(C^{i,j} \times B^i) = \{j\} \times B^j$ to B^j by the correspondence $(j, b) \mapsto b$.

LEMMA 2.4: *The map (C, h) is an identity morphism of the object B.*

Proof: Let $B_0 = (B_0^i \mid i \in I_0)$ be an object in the category $\mathbf{c}$ and let $(C_0, h_0) \in \mathbf{c}'(B_0, B)$. Denote by $(\widetilde{C}_0, \widetilde{h}_0)$ the composition $(C, h) \circ (C_0, h_0)$. We prove that

$$(\widetilde{C}_0, \widetilde{h}_0) \sim (C_0, h_0). \tag{2.5}$$

By the definition, for every $i \in I_0$, and $j \in I$,

$$\widetilde{C}_0^{i,j} = C^{j,j} \times C_0^{i,j} = \{j\} \times C_0^{i,j},$$

and for every $b \in B_0^i$ and $c \in C_0^{i,j}$,

$$\widetilde{h}_0^j((j, c), b) = h^j(j, h_0^j(c, b)) = h_0^j(c, b). \tag{2.6}$$

For all $i \in I_0$, $j \in I$, define a map $g^{i,j}\colon C_0^{i,j} \to \widetilde{C}_0^{i,j}$ by the correspondence $c \mapsto (j, c)$. It follows from (2.6) that for every $b \in B_0^i$ and $c \in C_0^{i,j}$,

$$h_0^j(c, b) = \widetilde{h}_0^j(g^{i,j}(c), b).$$

This proves (2.5).

On the other hand, let $B_1 = (B_1^j \mid j \in I_1)$ be an object of the category $\mathbf{c}$ and let $(C_1, h_1) \in \mathbf{c}'(B, B_1)$. Denote by $(\widetilde{C}_1, \widetilde{h}_1)$ the composition $(C_1, h_1) \circ (C, h)$. We prove that

$$(C_1, h_1) \sim (\widetilde{C}_1, \widetilde{h}_1). \tag{2.7}$$

Let $i \in I$, and $j \in I_1$. By the definition,

$$\widetilde{C}_1^{i,j} = C_1^{i,j} \times C^{i,i} = C_1^{i,j} \times \{i\},$$

and for every $b \in B^i$, $c \in C_1^{i,j}$,

$$\widetilde{h}_1^j((c, i), b) = h_1^j(c, h^i(i, b)) = h_1^j(c, b). \tag{2.8}$$

For all $c \in C_1^{i,j}$, define $g^{i,j}(c) = (c, i)$. Then, by (2.8), for every $b \in B^i$ and $c \in C_1^{i,j}$,

$$h_1^j(c, b) = \widetilde{h}_1^j(g^{i,j}(c), b).$$

This proves (2.7). ∎

Now we know that $\mathbf{c}$ is a category. The next step is to define a functor, which we shall denote by A, from the category $\mathbf{c}$ to the category $\mathbf{m}$ of matricial algebras. Let $B = (B_i \mid i \in I)$ be an object of the category $\mathbf{c}$. For all $i \in I$, denote by $V(B^i)$ the vector space with basis B^i, and let $V(B) = \bigoplus_{i \in I} V(B^i)$ be the vector space with basis B (note that since the sets B_i, $i \in I$, are disjoint, the family $(V(B_i) \mid i \in I)$ of vector spaces is independent). Define

$$A(B) = \{\alpha \in \mathrm{End}(V(B)) \mid \forall i \in I \colon \alpha(V(B^i)) \subseteq V(B^i)\}.$$

For all $\alpha \in \mathrm{End}(V(B))$, denote by α^i the restriction $\alpha \restriction V(B^i)$. Observe that $A(B)$ is a matricial algebra isomorphic to $\prod_{i \in I} \mathrm{End}(V(B^i))$.

Let $(C,h)\colon B_1 \to B_2$ be a premorphism in the category $\mathbf{c}$. For all $i \in I_1$, $j \in I_2$, denote by $V(C^{i,j})$ the vector space with basis $C^{i,j}$. For every $j \in I_2$, the bijection

$$h^j \colon \bigcup_{i \in I_1} (C^{i,j} \times B_1^i) \xrightarrow{\cong} B_2^j$$

induces an isomorphism

$$\phi^j \colon \bigoplus_{i \in I_1} (V(C^{i,j}) \otimes V(B_1^i)) \xrightarrow{\cong} V(B_2^j).$$

For all $\alpha \in A(B_1)$, set

$$A(C,h)(\alpha) = \bigoplus_{j \in I_2} \phi^j \circ \left(\bigoplus_{i \in I_1} (\mathbf{1}_{V(C^{i,j})} \otimes \alpha^i) \right) \circ (\phi^j)^{-1}. \tag{2.9}$$

Observe that $A(C,h)(\alpha)^j$ is an endomorphism of the vector space $V(B_2^j)$ for every $j \in I_2$, and so $A(C,h)(\alpha) \in A(B_2)$.

LEMMA 2.5: *Let B_1, B_2 be objects of the category $\mathbf{c}$ and let $(C,h) \in \mathbf{c}(B_1, B_2)$. Then $A(C,h)\colon A(B_1) \to A(B_2)$ is a homomorphism of unitary k-algebras.*

Proof: It suffices to verify that for every $\alpha, \beta \in A(B_1)$ and for every element t of the field k,

$$A(C,h)(\alpha + \beta) = A(C,h)(\alpha) + A(C,h)(\beta),$$
$$A(C,h)(\alpha \circ \beta) = A(C,h)(\alpha) \circ A(C,h)(\beta),$$
$$A(C,h)(t\alpha) = tA(C,h)(\alpha),$$

and

$$A(C,h)(\mathbf{1}_{V(B_1)}) = \mathbf{1}_{V(B_2)}.$$

But all these equalities are clear from the definition. ∎

LEMMA 2.6: *Let B_1, B_2 be objects of the category* $\mathbf{c}$ *and let* (C,h), $(\widetilde{C},\widetilde{h}) \in \mathbf{c}'(B_1,B_2)$. *If* $(C,h) \sim (\widetilde{C},\widetilde{h})$, *then* $A(C,h) = A(\widetilde{C},\widetilde{h})$.

Proof: Since $(C,h) \sim (\widetilde{C},\widetilde{h})$, there are bijections $g^{i,j}\colon C^{i,j} \xrightarrow{\cong} \widetilde{C}^{i,j}$ such that for every $b \in B_1^i$, $c \in C^{i,j}$,

$$\widetilde{h}^j(g^{i,j}(c),b) = h^j(c,b) \quad (i \in I_1,\ j \in I_2).$$

The bijections $g^{i,j}$ induce isomorphisms $\gamma^{i,j}\colon V(C^{i,j}) \to V(\widetilde{C}^{i,j})$ satisfying

$$\widetilde{\phi}^j \circ \left(\bigoplus_{i\in I_1} (\gamma^{i,j} \otimes \mathbf{1}_{V(B_1^i)}) \right) = \phi^j,$$

and

$$\left(\bigoplus_{i\in I_1} ({\gamma^{i,j}}^{-1} \otimes \mathbf{1}_{V(B_1^i)}) \right) \circ (\widetilde{\phi}^j)^{-1} = (\phi^j)^{-1}$$

for every $j \in I_2$. Substituting in (2.9), a straightforward computation leads to the equality $A(C,h)(\alpha) = A(\widetilde{C},\widetilde{h})(\alpha)$ for every $\alpha \in A(B_1)$. ∎

We define $A([C,h]) = A(C,h)$ for every morphism $[C,h] \in \mathbf{c}(B_1,B_2)$. In order to prove that A is a functor we have to verify that it preserves both the composition of morphisms and the identity morphisms.

LEMMA 2.7: *The functor A preserves the composition of morphisms. In particular, let* $B_n = (B_n^i \mid i \in I_n)$, $n = 1,2,3$, *be objects of the category* $\mathbf{c}$ *and let* $(C_1,h_1) \in \mathbf{c}'(B_1,B_2)$, $(C_2,h_2) \in \mathbf{c}'(B_2,B_3)$ *be premorphisms. Then*

$$A((C_2,h_2)\circ(C_1,h_1)) = A(C_2,h_2)\circ A(C_1,h_1).$$

Proof: Denote by (C,h) the composition $(C_2,h_2)\circ(C_1,h_1)$. Recall that for every $i \in I_1$, $k \in I_3$,

$$C^{i,k} = \bigcup_{j\in I_2} (C_2^{j,k} \times C_1^{i,j})$$

and for every $b \in B_1^i$, $c_1 \in C_1^{i,j}$, and $c_2 \in C_2^{j,k}$, where $i \in I_1$, $j \in I_2$ and $k \in I_3$,

$$h^k((c_2,c_1),b) = h_2^k(c_2, h_1^j(c_1,b)).$$

It follows that

$$\phi^k((c_2\otimes c_1)\otimes b) = \phi_2^k(c_2 \otimes \phi_1^j(c_1\otimes b)),$$

where ϕ_1^j, ϕ_2^k, ϕ^k are the vector space isomorphisms induced by the maps h_1^j, h_2^k, h^k, respectively. Thus, for every $k \in I_3$,

$$\phi^k = \phi_2^k \circ \left(\bigoplus_{j \in I_2} (\mathbf{1}_{V(C_2^{j,k})} \otimes \phi_1^j) \right) \circ \theta^k,$$

where θ^k is the "corrective" homomorphism induced by the correspondence

$$(c_2 \otimes c_1) \otimes b \mapsto c_2 \otimes (c_1 \otimes b)$$

(here again $b \in B_1^i$, $c_1 \in C_1^{i,j}$, $c_2 \in C_2^{j,k}$).

Let $k \in I_3$. Put $\psi_1^k = (\bigoplus_{j \in I_2} (\mathbf{1}_{V(C_2^{j,k})} \otimes \phi_1^j)) \circ \theta^k$, and compute that for every $\alpha \in A(B_1)$,

$$\psi_1^k \circ \left(\bigoplus_{i \in I_1} (\mathbf{1}_{V(C^{i,k})} \otimes \alpha^i) \right) \circ \psi_1^{k^{-1}} = \bigoplus_{j \in I_2} (\mathbf{1}_{V(C_2^{j,k})} \otimes A(C_1, h_1)(\alpha)^j). \tag{2.10}$$

Composing the morphisms in equality (2.10) with ϕ_2^k, resp. $(\phi_2^k)^{-1}$ from the left, resp. right hand side, we get that

$$A(C,h)(\alpha)^k = A(C_2, h_2)(A(C_1, h_1)(\alpha))^k. \qquad \blacksquare$$

LEMMA 2.8: *Let $B = (B^i \mid i \in I)$ be an object of the category $\mathbf{c}$. If $[C,h]$ is the identity morphism on B, then $A(C,h) = \mathbf{1}_{A(B)}$.*

Proof: Let $B_1 = (B_1^j \mid j \in I_1)$ be an object of the category $\mathbf{c}$ and $(C_1, h_1) \in \mathbf{c}'(B, B_1)$ a premorphism such that $C_1^{i,j} \neq \emptyset$ for every $i \in I$, $j \in I_1$. Then the homomorphism $A(C_1, h_1)$ is one-to-one, and by Lemmas 2.4, 2.6 and 2.7,

$$A(C_1, h_1) \circ A(C, h) = A((C_1, h_1) \circ (C, h)) = A(C_1, h_1).$$

It follows that $A(C,h) = \mathbf{1}_{A(B)}$. $\blacksquare$

We define a functor $\Lambda \colon \mathbf{c} \to \mathbf{b}$ as follows: For each object $B = (B^i \mid i \in I)$, we define $\Lambda(B)$ to be the power-set semilattice $\mathcal{P}(I)$ of the set I. Given a premorphism $(C,h) \in \mathbf{c}'(B_1, B_2)$, we define a $(\vee, 0, 1)$-semilattice homomorphism $\Lambda(C,h) \colon \Lambda(B_1) \to \Lambda(B_2)$ by the rule

$$J \mapsto \left\{ j \in I_2 \mid \bigcup_{i \in J} C^{i,j} \neq \emptyset \right\} \quad (J \in \mathcal{P}(I_1)).$$

It is clear that $(C,h) \sim (\widetilde{C}, \widetilde{h})$ implies that $\Lambda(C,h) = \Lambda(\widetilde{C}, \widetilde{h})$. Thus we are entitled to define $\Lambda([C,h]) = \Lambda(C,h)$.

Any two-sided ideal of a matricial algebra is principal. For every $\alpha \in A(B)$, we denote by $\langle \alpha \rangle$ the two-sided ideal generated by the homomorphism α. Then the rule

$$\langle \alpha \rangle \mapsto \{i \in I \mid \alpha^i \neq 0\}$$

defines an isomorphism $\eta_B\colon \operatorname{Id}^c A(B) \to \Lambda(B)$.

LEMMA 2.9: *The isomorphism $\eta\colon \operatorname{Id}^c A \to \Lambda$ is natural.*

Proof: We prove that for every $(C,h) \in \mathbf{c}'(B_1,B_2)$, the diagram

$$\begin{array}{ccc} \operatorname{Id}^c A(B_1) & \xrightarrow{\operatorname{Id}^c A(C,h)} & \operatorname{Id}^c A(B_2) \\ \downarrow{\scriptstyle \eta_{B_1}} & & \downarrow{\scriptstyle \eta_{B_2}} \\ \Lambda(B_1) & \xrightarrow[\Lambda(C,h)]{} & \Lambda(B_2) \end{array}$$

commutes. Let $j \in I_2$ and $\alpha \in A(B_1)$. Then

$$\Lambda(C,h) \circ \eta_{B_1}(\langle \alpha \rangle) = \{j \in I_2 \mid \exists i \in I_1\colon \alpha^i \neq 0 \ \&\ C^{i,j} \neq \emptyset\}.$$

Set $\beta = A(C,h)(\alpha)$. Then

$$\eta_{B_2} \circ \operatorname{Id}^c A(C,h)(\langle \alpha \rangle) = \eta_{B_2}(\langle \beta \rangle) = \{j \in I_2 \mid \beta^j \neq 0\}$$

and, by the definition, for every $j \in I_2$,

$$\beta^j = \phi^j \circ \left(\bigoplus_{i \in I_1} (\mathbf{1}_{V(C^{i,j})} \otimes \alpha^i) \right) \circ {\phi^j}^{-1},$$

where ϕ^j is the isomorphism induced by the bijection h^j. Then $\beta^j \neq 0$ iff

$$\bigoplus_{i \in I_1} (\mathbf{1}_{V(C^{i,j})} \otimes \alpha^i) \neq 0$$

iff there is $i \in I_1$ such that $\alpha^i \neq 0$ and $C^{i,j} \neq \emptyset$. ∎

Definition: Let $f\colon s_1 \to s_2$ be a homomorphism in $\mathbf{s}$. Let B_1, B_2 be objects of the category B and let $\varepsilon_i\colon I_i \to J(s_i)$, $i = 1,2$, be isomorphisms of posets. We say that a morphism $[C,h] \in \mathbf{c}(B_1,B_2)$ is **f-induced with respect to ε_1, ε_2** if the diagram

$$\begin{array}{ccc} Bo(s_1) & \xrightarrow{Bo(f)} & Bo(s_2) \\ \downarrow{\scriptstyle \mathcal{P}(\varepsilon_1)} & & \downarrow{\scriptstyle \mathcal{P}(\varepsilon_2)} \\ \Lambda(B_1) & \xrightarrow{\Lambda([C,h])} & \Lambda(B_2) \end{array}$$

commutes.

Observe that the morphism $[C, h]$ is f-induced with respect to ε_1, ε_2 if and only if $C^{i,j} \neq 0$ iff $f(\varepsilon_1(i)) \geq \varepsilon_2(j)$ for every $i \in I_1$, $j \in I_2$.

PROPOSITION 2.10: *Let P be a partially ordered upwards directed set without maximal elements. Let*

$$\langle s_p, f_{p,q} \rangle_{p \leq q \text{ in } P}$$

be a direct system in **s**. *Let*

$$\langle B_p, [C_{p,q}, h_{p,q}] \rangle_{p<q \text{ in } P}$$

be a direct system in the category **c** *and $(\varepsilon_p\colon I_p \to J(s_p) \mid p \in P)$ a family of bijections such that $[C_{p,q}, h_{p,q}]$ is a $f_{p,q}$-induced morphism with respect to ε_p, ε_q for every $p < q$ in P. If R is a direct limit of the diagram*

$$\langle A(B_p), A([C_{p,q}, h_{p,q}]) \rangle_{p<q \text{ in } P},$$

then $\mathrm{Id}^{\mathrm{c}}(R)$ is isomorphic to $\varinjlim \langle s_p, f_{p,q} \rangle_{p \leq q \text{ in } P}$.

Proof: This follows from Proposition 1.1 and the fact that the functor Id^{c} commutes with direct limits. ∎

3. Bergman's theorems

The purpose of this section is to illustrate the effectiveness of the tools developed in Sections 1 and 2. The results proved here are not going to be used later in the paper. We reprove the two main results from the unpublished notes by G. M. Bergman [Be]. Different proofs of the first of them were published in [GW]. It states that every countable distributive $(\vee, 0, 1)$-semilattice is isomorphic to the semilattice of finitely generated two-sided ideals of a locally matricial algebra. As far as I know, the second theorem has never been published. It is the following assertion: Every strongly distributive $(\vee, 0, 1)$-semilattice is isomorphic to the semilattice of finitely generated ideals of a locally matricial algebra. A $(\vee, 0)$-semilattice is **strongly distributive** provided every element is a join of join irreducible elements. The ideal lattices of strongly distributive $(\vee, 0)$-semilattices are characterized as the lattices of all hereditary subsets of partially ordered sets [Be]. A strongly distributive $(\vee, 0)$-semilattice has a unit element if and only if the corresponding partially ordered set P has finitely many maximal elements and every element of P is under one of them [Be].

THEOREM 3.1: *Every countable distributive* $(\vee, 0, 1)$*-semilattice is isomorphic to the semilattice of finitely generated two-sided ideals of a unital locally matricial algebra.*

Proof: Let S be a countable distributive $(\vee, 0, 1)$-semilattice. By a theorem of P. Pudlák, the semilattice S is the directed union of its finite distributive $(\vee, 0, 1)$-subsemilattices [Pu]. Since S is countable, there is a countable sequence

$$s_0 \subseteq s_1 \subseteq s_2 \subseteq \cdots$$

of finite $(\vee, 0, 1)$-semilattices such that $S = \bigcup_{i \in \omega} s_i$. Put $I_n = J(s_n)$ and, for all $n \leq m$ in ω, denote by $f_{n,m}$ the inclusion map $s_n \to s_m$.

For each $n \in \omega$ and $i \in I_n$, put

$$B_n^i = \{(i_0, \ldots, i_n) \in I_0 \times \cdots \times I_n \mid i_0 \geq \cdots \geq i_n = i\}.$$

Given $n < m$ in ω, set

$$C_{n,m}^{i,j} = \{(i_n, \ldots, i_m) \in I_n \times \cdots \times I_m \mid i = i_n \geq \cdots \geq i_m = j\} \quad (i \in I_n,\ j \in I_m)$$

and for every $j \in I_m$, define an isomorphism $h_{n,m}^j \colon \bigcup_{i \in I_n} (C_{n,m}^{i,j} \times B_n^i) \to B_m^j$ by the rule

$$((i_n, \ldots, i_m), (i_0, \ldots, i_n)) \mapsto (i_0, \ldots, i_m).$$

We verify that

(i) for every $n \in \omega$, for every $i \in I_n$, $B_n^i \neq 0$,

(ii) if $n < m$, then for every $i \in I_n$, $j \in I_m$, $C_{n,m}^{i,j} \neq 0$ iff $i \geq j$.

Ad (i): Let $n \in \omega$. It suffices to prove that for every $i \in I_{n+1}$ there exists $j \geq i$ in I_n. Since $\bigvee I_n = 1 \geq i$ and i is join irreducible, there is $j \in I_n$ with $j \geq i$ and we are done.

Ad (ii): Let $n < m$ in ω. Let $i \in I_n$ and $j \in I_m$ satisfy $i \geq j$. Then there exist $k_0, \ldots, k_{t-1} \in I_{n+1}$ with $i = k_0 \vee \cdots \vee k_{t-1}$, and since $i \geq j$ and j is join irreducible, $k_s \geq j$ for some $s < t$. Thus $i \geq k \geq j$ for some $k \in I_{n+1}$. By induction we prove that if $i \geq j$, then $C_{n,m}^{i,j} \neq 0$. The converse implication is clear from the definition.

Having verified (i), it is clear that

$$\langle B_n, [C_{n,m}, h_{n,m}] \rangle_{n<m \text{ in } \omega}$$

is a direct system in $\mathbf{c}$. It follows from (ii) that for every $n < m$ in ω, $\Lambda([C_{n,m}, h_{n,m}]) = Bo(f_{n,m})$, that is, $[C_{n,m}, h_{n,m}]$ is an $f_{n,m}$-induced morphism with respect to identity maps. Now we apply Proposition 2.10. ∎

THEOREM 3.2: *Every strongly distributive* $(\vee, 0, 1)$*-semilattice is isomorphic to the semilattice of finitely generated ideals of a unital locally matricial algebra.*

Proof: Let S be a strongly distributive $(\vee, 0)$-semilattice. Then there is a partially ordered set Q such that S is isomorphic to the semilattice of compact elements of the lattice $H(Q)$, that is,

$$S \simeq \{(F] \mid F \in [Q]^{<\omega}\}.$$

The semilattice S has a greatest element if and only if $Q = (M]$ for some finite subset M of Q (i.e., if for every $q \in Q$ there is $m \in M$ with $q \leq m$). Put

$$K = \{F \in [Q]^{<\omega} \mid M \subseteq F\}$$

and $P = K \times \omega$. Define an order relation on the set P by $(I, n) < (J, m)$ if $I \subseteq J$ and $n < m$. Observe that P is upwards directed without maximal elements.

Given a pair $p = (I_p, n) \leq q = (I_q, m)$ in P, let $f_{p,q}\colon H(I_p) \to H(I_q)$ denote the semilattice homomorphism given by $f_{p,q}((i]_{I_p}) = (i]_{I_q}$ for every $i \in I_p$. The homomorphism $f_{p,q}$ preserves 0 and 1 and

$$S = \varinjlim \langle H(I_p), f_{p,q}\rangle_{p \leq q \text{ in } P}.$$

Let $p = (I_p, n) \in P$. For each $i \in I_p$, let B_p^i be the set of pairs $(\underline{n}, \underline{i})$, where $\underline{n} = (n_1, \ldots, n_s)$ is a sequence of natural numbers not bigger than n and $\underline{i} = (i_0, \ldots, i_s)$ is a sequence of elements of I_p such that $i_0 \in M$ and $i_0 > \cdots > i_s = i$ (s is a natural number). It is clear that the set B_p^i is nonempty for every $i \in I_p$.

Let $p = (I_p, n) < q = (I_q, m)$ be a pair of elements of P. Given $i \in I_p$ and $j \in I_q$, we define $C_{p,q}^{i,j}$ to be the set of pairs $(\underline{m}, \underline{j})$ such that $\underline{m} = (m_1, \ldots, m_t)$ is a sequence of natural numbers not bigger than m and $\underline{j} = (j_0, \ldots, j_t)$ is a sequence of elements of I_q satisfying $i = j_0 > \cdots > j_t = j$ (t is a natural number) and if $i > j$, then either $m_1 > n$ or $j_1 \notin I_p$.

Given pairs $(\underline{n}', \underline{i}') \in B_p^i$, where $\underline{n}' = (n_1, \ldots, n_s)$ and $\underline{i}' = (i_0, \ldots, i_s)$, and $(\underline{n}'', \underline{i}'') \in C_{p,q}^{i,j}$, where $\underline{n}'' = (n_{s+1}, \ldots, n_t)$ and $\underline{i}'' = (i_s, \ldots, i_t)$, we define

$$h_{p,q}^j((\underline{n}'', \underline{i}''), (\underline{n}', \underline{i}')) = (\underline{n}, \underline{i}),$$

where $\underline{n} = (n_1, \ldots, n_t)$ and $\underline{i} = (i_0, \ldots, i_t)$. It is readily seen that $(\underline{n}, \underline{i}) \in B_q^j$, and so we have defined a map $h_{p,q}^j\colon \bigcup_{i \in I_p}(C_{p,q}^{i,j} \times B_p^i) \to B_q^j$. On the other hand, let $(\underline{n}, \underline{i})$, where $\underline{n} = (n_1, \ldots, n_t)$ and $\underline{i} = (i_0, \ldots, i_t)$, be an element of B_q^j.

Denote by s the maximal number from the set $\{0,\dots,t\}$ such that $i_s \in I_p$ and the pair $(\underline{n}',\underline{i}')$, where $\underline{n}' = (n_1,\dots,n_s)$ and $\underline{i}' = (i_0,\dots,i_s)$, belongs to $B_p^{i_s}$. If $s=t$, let $\underline{n}''$ be an empty sequence and $\underline{i}'' = (i_t)$, while if $s<t$, define $\underline{n}'' = (n_{s+1},\dots,n_t)$ and $\underline{i}'' = (i_s,\dots,i_t)$. It follows from the choice of s that if $s<t$, then either $n_{s+1} > n$ or $i_{s+1} \notin I_p$. Hence $(\underline{n}'',\underline{i}'') \in C_{p,q}^{i,j}$ and the correspondence $(\underline{n},\underline{i}) \mapsto ((\underline{n}'',\underline{i}''),(\underline{n}',\underline{i}'))$ defines a map $h'^{j}_{p,q}\colon B_q^j \to \bigcup_{i\in I_p}(C_{p,q}^{i,j}\times B_p^i)$. The map $h'^{j}_{p,q}$ is clearly one-to-one and the composition $h'^{j}_{p,q}\circ h^j_{p,q}$ equals the identity map on the set $\bigcup_{i\in I_p}(C_{p,q}^{i,j}\times B_p^i)$. It follows that the map $h^j_{p,q}$ is a bijection.

Let $p=(I_p,n) < q = (I_q,m) < r = (I_r,l)$ be elements of P, let $i\in I_p$, $j\in I_q$ and $k\in I_r$. For all $(\underline{m}',\underline{j}') \in C_{p,q}^{i,j}$, where $\underline{m}' = (m_1,\dots,m_s)$ and $\underline{j}' = (j_0,\dots,j_s)$, and $(\underline{m}'',\underline{j}'') \in C_{q,r}^{j,k}$, where $\underline{m}'' = (m_{s+1},\dots,m_t)$, $\underline{j}'' = (j_s,\dots,j_t)$, define

$$g^{i,k}_{p,q,r}((\underline{m}'',\underline{j}''),(\underline{m}',\underline{j}')) = (\underline{m},\underline{j}),$$

where $\underline{m} = (m_1,\dots,m_t)$ and $\underline{j} = (j_0,\dots,j_t)$. Notice that $g^{i,k}_{p,q,r}$ is a map from $\bigcup_{j\in I_q}(C_{q,r}^{j,k}\times C_{p,q}^{i,j})$ to $C_{p,r}^{i,k}$. Let $i\in I_p$, $j\in I_q$ and $k\in I_r$ satisfy $i\geq j\geq k$. Then for every natural number $s\leq t\leq u$, and $(\underline{n},\underline{i})\in B_p^i$, where $\underline{n}=(n_1,\dots,n_s)$, $\underline{i} = (i_0,\dots,i_s)$, $(\underline{m}',\underline{j}') \in C_{p,q}^{i,j}$, where $\underline{m}' = (m_{s+1},\dots,m_t)$, $\underline{j}' = (j_s,\dots,j_t)$, and $(\underline{m}'',\underline{j}'') \in C_{q,r}^{j,k}$, where $\underline{m}'' = (m_{t+1},\dots,m_u)$, $\underline{j}'' = (j_t,\dots,j_u)$,

$$\begin{aligned} h^k_{p,r}(g^{i,k}_{p,q,r}((\underline{m}'',\underline{j}''),(\underline{m}',\underline{j}')),(\underline{n},\underline{i})) &= (\underline{m},\underline{j}) \\ &= h^k_{q,r}((\underline{m}'',\underline{j}''),h^j_{p,q}((\underline{m}',\underline{j}'),(\underline{n},\underline{i}))), \end{aligned}$$

where $\underline{m} = (n_1,\dots,n_s,m_{s+1},\dots,m_u)$, and $\underline{j} = (i_0,\dots,i_s,j_{s+1},\dots,j_n)$. (Note that $i_s = j = j_s$.) It follows that

$$\langle B_p,[C_{p,q},h_{p,q}]\rangle_{p<q \text{ in } P}$$

forms a direct system in the category $\mathbf{c}$. For every $p\in P$ define a bijection $\varepsilon_p\colon I_p\to J(H(I_p))$ by $i\mapsto (i]_{I_p}$. It is clear that given $p=(I_n,n) < q = (I_q,m)$ in P, for every $i\in I_p$, $j\in I_q$, the inequality $i\geq j$ (i.e., $(i]_{I_q}\supseteq (j]_{I_q}$) holds iff $C_{p,q}^{j,i}\neq\emptyset$, whence the morphism $[C_{p,q},h_{p,q}]$ is $f_{p,q}$-induced with respect to ε_p, ε_q. Proposition 2.10 concludes the proof. ∎

4. Representation of distributive lattices

Let M be a finite set. Denote by $TO(M)$ the set of all total orders on the set M. For all $\alpha\in TO(M)$, denote by $H(\alpha)$ the set of all hereditary subsets (including the empty set) of M with respect to the order α.

Let N be a subset of a finite set M and let $\alpha \in TO(M)$. Denote by $\alpha \restriction N$ the restriction of α to the set N. For all α: $a_0 < \cdots < a_n$ and β: $b_0 < \cdots < b_n \in TO(M)$ define $\alpha \sim_N \beta$ if $a_i \neq b_i$ implies $a_i, b_i \in N$ for every $i \in \{0, \dots, n\}$. It is clear that $\sim_N$ is an equivalence relation on the set $TO(M)$, and we denote by $[\alpha]_N$ the equivalence class of the linear order α.

Lemma 4.1: *Let N be a subset of a finite set M. For every $\alpha \in TO(N)$ and $\gamma \in TO(M)$, there exists a unique $\beta \in TO(M)$ satisfying $\beta \sim_N \gamma$ and $\beta \restriction N = \alpha$.*

Proof: For β, $\gamma \in N$, $\beta \sim_N \gamma$ iff there exists a permutation σ of M fixing every element of $M \setminus N$ such that $a <_\beta b$ iff $\sigma(a) <_\gamma \sigma(b)$, for all $a, b \in M$. The conclusion easily follows. ∎

Let Q be a subset of the set $\mathcal{P}(M)$. Denote by $C(Q)$ the set

$$\{\varphi: Q \to \mathcal{P}(M) \mid \forall N \in Q: \varphi(N) \subseteq N\}.$$

For every $\varphi \in C(Q)$, put

$$\cup\varphi = \bigcup\{\varphi(N) \mid N \in Q\}.$$

Definition: Let L be a finite distributive lattice. For all $a \in J(L)$, let B_L^a be the set of all pairs (α, φ), where $\alpha \in TO([a)_L)$, $\varphi \in C(\mathcal{P}(L))$, and the following properties are satisfied:

(i) $[a)_L \supseteq \cup\varphi$,

(ii) for all $a' > a$ in $J(L)$, if $[a')_L \in H(\alpha)$, then $[a')_L \not\supseteq \cup\varphi$.

Denote by B_L the family $(B_L^a \mid a \in J(L))$; it is an object of $\mathbf{b}$ associated to the finite distributive lattice L.

Let L_1 be a $(0,1)$-sublattice of a finite distributive lattice L_2. Let $a \in J(L_1)$ and $b \in J(L_2)$. If $b \not\leq a$, then we put $C_{L_1,L_2}^{a,b} = \emptyset$. Suppose that $b \leq a$, that is, $[b)_{L_2} \supseteq [a)_{L_1}$. Then we define $C_{L_1,L_2}^{a,b}$ to be the set of all pairs $([\beta']_{[a)_{L_1}}, \psi')$, where $\beta' \in TO([b)_{L_2})$, $\psi' \in C(\mathcal{P}(L_2) \setminus \mathcal{P}(L_1))$, and the following properties are satisfied:

(iii) $[a)_{L_1} \in H(\beta' \restriction ([b)_{L_2} \cap L_1))$,

(iv) $[b)_{L_2} \supseteq \cup\psi'$,

(v) for all $b' \in J(L_2)$ with $b < b' \leq a$, if $[b')_{L_2} \in H(\beta')$, then $[b')_{L_2} \not\supseteq \cup\psi'$.

(Observe that if $\beta \sim_{[a)_{L_1}} \beta'$, then $[a)_{L_1} \in H(\beta' \restriction ([b)_{L_2} \cap L_1))$ iff $[a)_{L_1} \in H(\beta \restriction ([b)_{L_2} \cap L_1))$ and for every $b' \in J(L_2)$ with $b < b' \leq a$, $[b')_{L_2} \in H(\beta)$ iff $[b')_{L_2} \in H(\beta')$; hence the definition is correct.) The following lemma is well-known [MMT, Exercises 2.63.10].

LEMMA 4.2: *Let L_1 be a $(0,1)$-sublattice of a finite distributive lattice L_2. Then for every $b \in J(L_2)$, $[b)_{L_2} \cap L_1 = [c)_{L_1}$ for some $c \in J(L_1)$.*

LEMMA 4.3: *Let L_1 be a $(0,1)$-sublattice of a finite distributive lattice L_2. Let $b \in J(L_2)$. The rule*

$$(([\beta']_{[a)_{L_1}}, \psi'), (\alpha, \varphi)) \mapsto (\beta, \psi), \tag{4.2}$$

where $\psi = \psi' \cup \varphi$ and $\beta \in TO([b)_{L_2})$ satisfies $\beta \sim_{[a)_{L_1}} \beta'$ and $\beta \restriction [a)_{L_1} = \alpha$, defines a map

$$h^b_{L_1,L_2}\colon \bigcup_{a \in J(L_1)} (C^{a,b}_{L_1,L_2} \times B^a_{L_1}) \to B^b_{L_2}.$$

Proof: Let $a \in J(L_1)$. If $b \not\leq a$, then the set $C^{a,b}_{L_1,L_2}$ is empty. Suppose that $b \leq a$. Let $(\alpha, \varphi) \in B^a_{L_1}$, and $([\beta']_{[a)_{L_1}}, \psi') \in C^b_{L_1,L_2}$. Let (β, ψ) be the pair defined by the correspondence (4.2). According to Lemma 4.1 such a pair exists and is uniquely determined. We prove that $(\beta, \psi) \in B^b_{L_2}$. It suffices to verify that

(i) $[b)_{L_2} \supseteq \cup\psi$,

(ii) for all $b' > b$ in $J(L_2)$, if $[b')_{L_2} \in H(\beta)$, then $[b')_{L_2} \not\supseteq \cup\psi$.

Ad (i): By the definition $[b)_{L_2} \supseteq \cup\psi'$. Since we have supposed that $b \leq a$, $[b)_{L_2} \supseteq [a)_{L_1} \supseteq \cup\varphi$. It follows that $[b)_{L_2} \supseteq (\cup\psi') \cup (\cup\varphi) = \cup\psi$.

Ad (ii): Let $[b')_{L_2} \in H(\beta)$ for some $b \leq b' \in J(L_2)$. If $b' \not\supseteq \cup\psi'$ we are done. Assume otherwise. Then, by property (v) of $C^{a,b}_{L_1,L_2}$, $b' \not\leq a$, that is, $[b')_{L_2} \cap L_1 \not\supseteq [a)_{L_1}$. By Lemma 4.2, $[b')_{L_2} \cap L_1 = [a')_{L_1}$ for some $a' \in J(L_1)$. Since $[b')_{L_2} \in H(\beta)$, we have that $[a')_{L_1} \in H(\beta \restriction ([b)_{L_2} \cap L_1)))$. By property (iii) of $C^{a,b}_{L_1,L_2}$, also $[a)_{L_1} \restriction H(\beta \in ([b)_{L_2} \cap L_1)))$, and so either $[a')_{L_1} \supseteq [a)_{L_1}$ or $[a)_{L_1} \supsetneq [a')_{L_1}$. According to the assumption that $b' \not\leq a$, only the latter case is possible, and so $a < a'$ and $[a')_{L_1} \in H(\alpha)$. By property (ii) of $B^a_{L_1}$, we have that $[a')_{L_1} \not\supseteq \cup\varphi$, whence $[b')_{L_2} \not\supseteq \cup\psi$. ∎

LEMMA 4.4: *Let L_1 be a $(0,1)$-sublattice of a finite distributive lattice L_2. Let $b \in J(L_2)$. The map $h^b_{L_1,L_2}$ defined by (4.2) is a bijection.*

Proof: First we prove that the map $h^b_{L_1,L_2}$ is onto. Let $(\beta, \psi) \in B^b_{L_2}$. Denote by φ the restriction $\psi \restriction \mathcal{P}(L_1)$. By Lemma 4.2, $[b)_{L_2} \cap L_1 = [c)_{L_1}$ for some $c \in J(L_1)$. Since, by property (i) of $B^b_{L_2}$, $[b)_{L_2} \supseteq \cup\psi$, we have that $[c)_{L_1} \supseteq \cup\varphi$. The set of all $a' \in J(L_1)$ for which $[a')_{L_1} \in H(\beta \restriction ([b)_{L_2} \cap L_1))$ and $[a')_{L_1} \supseteq \cup\varphi$ is nonempty (it contains at least c) and totally ordered with respect to β. Let

a be the greatest element of this set. Put $\alpha = \beta \restriction [a)_{L_1}$. It is straightforward that $(\alpha, \varphi) \in B^a_{L_1}$.

Denote by ψ' the restriction $\psi \restriction (\mathcal{P}(L_2) \setminus \mathcal{P}(L_1))$. Trivially $[b)_{L_2} \supseteq \bigcup\psi'$, and we have chosen $a \in L_1$ so that $[a)_{L_1} \in H(\beta \restriction ([b)_{L_2} \cap L_1))$. In order to prove that $([\beta]_{[a)_{L_2}}, \psi') \in C^{a,b}_{L_1,L_2}$, it suffices to verify that $[b')_{L_2} \not\supseteq \bigcup\psi'$ for every $b' \in J(L_2)$ such that $b < b' \leq a$ and $[b')_{L_2} \in H(\beta)$. Let $b' \in J(L_2)$ be any such element. Then $[b')_{L_2} \not\supseteq \bigcup\psi$ by property (iii) of $B^b_{L_2}$, and since $b' \leq a$ and $[a)_{L_1} \supseteq \bigcup\varphi$, we have that $[b')_{L_2} \supseteq [a)_{L_1} \supseteq \bigcup\varphi$, whence $[b')_{L_2} \not\supseteq \bigcup\psi'$.

By the definition,

$$h^b_{L_1,L_2}(([\beta]_{[a)_{L_1}}, \psi'), (\alpha, \varphi)) = (\beta, \psi).$$

It remains to verify that the map $h^b_{L_1,L_2}$ is one-to-one. Let

$$h^b_{L_1,L_2}(([\beta']_{[a)_{L_1}}, \psi'), (\alpha, \varphi)) = (\beta, \psi)$$

for some $a \in J(L_1)$, $([\beta']_{[a)_{L_2}}, \psi') \in C^{a,b}_{L_1,L_2}$, and $(\alpha, \varphi) \in B^a_{L_1}$. According to property (iii) of $C^{a,b}_{L_1,L_2}$, $[a)_{L_1} \in H(\beta' \restriction ([b)_{L_2} \cap L_1))$ which is equivalent to $[a)_{L_1} \in H(\beta \restriction ([b)_{L_2} \cap L_1))$. By property (ii) of $B^a_{L_1}$, $[a')_{L_1} \not\supseteq \bigcup\varphi$ for every $a < a' \in J(L_1)$ such that $[a')_{L_1} \in H(\alpha)$. Since $\alpha = \beta \restriction [a)_{L_1}$, a is the greatest element, with respect to the total order β, of the set of all $a' \in J(L_1)$ which satisfy $[a')_{L_1} \in H(\beta \restriction ([b)_{L_2} \cap L_1))$ and $[a')_{L_1} \supseteq \bigcup\varphi$. It follows that a is uniquely determined by the pair (β, ψ). Since $\varphi = \psi \restriction \mathcal{P}(L_1)$, $\alpha = \beta \restriction [a)_{L_1}$, $\psi' = \psi \restriction (\mathcal{P}(L_2) \setminus \mathcal{P}(L_1))$, and $[\beta']_{[a)_{L_1}} = [\beta]_{[a)_{L_1}}$, the map $h^b_{L_1,L_2}$ is one-to-one. ∎

LEMMA 4.5: *Let L_1 be a $(0,1)$-sublattice of a finite distributive lattice L_2, let L_2 be a $(0,1)$-sublattice of a finite distributive lattice L_3. Then*

$$[C_{L_1,L_3}, h_{L_1,L_3}] = [C_{L_2,L_3}, h_{L_2,L_3}] \circ [C_{L_1,L_2}, h_{L_1,L_2}].$$

Proof: Let $a \in J(L_1)$ and $c \in J(L_3)$. We set

$$\widetilde{C}^{a,c}_{L_1,L_2,L_3} = \bigcup_{b \in J(L_2)} (C^{b,c}_{L_2,L_3} \times C^{a,b}_{L_1,L_2}),$$

and we define a map $\widetilde{h}^c_{L_1,L_2,L_3}\colon \bigcup_{a \in J(L_1)} (\widetilde{C}^{a,c}_{L_1,L_2,L_3} \times B^a_{L_1}) \to B^c_{L_3}$ by the rule

$$\widetilde{h}^c_{L_1,L_2,L_3}((([\gamma']_{[b)_{L_2}}, \chi'), ([\beta']_{[a)_{L_1}}, \psi')), (\alpha, \varphi)) =$$
$$h^c_{L_2,L_3}(([\gamma']_{[b)_{L_2}}, \chi'), h^b_{L_1,L_2}(([\beta']_{[a)_{L_1}}, \psi'), (\alpha, \varphi)))$$

for every $(\alpha,\varphi) \in B^a_{L_1}$, $([\beta']_{[a)_{L_1}},\psi') \in C^{a,b}_{L_1,L_2}$, and $([\gamma']_{[b)_{L_2}},\chi') \in C^{b,c}_{L_2,L_3}$. By the definition of the composition of morphisms in the category **c**,

$$[\widetilde{C}_{L_1,L_2,L_3},\widetilde{h}_{L_1,L_2,L_3}] = [C_{L_2,L_3},h_{L_2,L_3}] \circ [C_{L_1,L_2},h_{L_1,L_2}].$$

For every $a \in J(L_1)$ and $c \in J(L_3)$, define a map $g^{a,c}_{L_1,L_2,L_3}\colon \widetilde{C}^{a,c}_{L_1,L_2,L_3} \to C^{a,c}_{L_1,L_3}$ by the rule

$$(([\gamma']_{[b)_{L_2}},\chi'),([\beta']_{[a)_{L_1}},\psi')) \mapsto ([\gamma'']_{[a)_{L_1}},\chi''),$$

where $\chi'' = \chi' \cup \psi'$ and γ'' satisfies both $\gamma'' \sim_{[b)_{L_2}} \gamma'$ and $(\gamma'' \restriction [b)_{L_2}) \sim_{[a)_{L_1}} \beta'$. By an argument similar to the one of the proof of Lemma 4.1, we easily see that such a $\gamma'' \in TO([c)_{L_3})$ exists and that its properties uniquely determine the equivalence class $[\gamma'']_{[a)_{L_1}}$.

Let $(\alpha,\varphi) \in B^a_{L_1}$, $([\beta']_{[a)_{L_1}},\psi') \in C^{a,b}_{L_1,L_2}$, and $([\gamma']_{[b)_{L_2}},\chi') \in C^{b,c}_{L_2,L_3}$. Let

$$([\gamma'']_{[a)_{L_1}},\chi'') = g^{a,c}_{L_1,L_2,L_3}(([\gamma']_{[b)_{L_2}},\chi'),([\beta']_{[a)_{L_1}},\psi')).$$

Then, on the one hand,

$$\begin{aligned}
&\widetilde{h}^c_{L_1,L_2,L_3}(((([\gamma']_{[b)_{L_2}},\chi'),([\beta']_{[a)_{L_1}},\psi')),(\alpha,\varphi))\\
&= h^c_{L_2,L_3}(([\gamma']_{[b)_{L_2}},\chi'),h^b_{L_1,L_2}(([\beta']_{[a)_{L_1}},\psi')),(\alpha,\varphi))\\
&= h^c_{L_2,L_3}(([\gamma']_{[b)_{L_2}},\chi'),(\beta,\psi)),
\end{aligned}$$

where $\psi = \psi' \cup \varphi$, $\beta \sim_{[a)_{L_1}} \beta'$, and $\beta \restriction [a)_{L_1} = \alpha$. Consequently,

$$h^c_{L_2,L_3}(([\gamma']_{[b)_{L_2}},\chi'),(\beta,\psi)) = (\gamma,\chi),$$

where $\chi = \chi' \cup \psi$, $\gamma \sim_{[b)_{L_2}} \gamma'$, and $\gamma \restriction [b)_{L_2} = \beta$, which implies both $(\gamma \restriction [b)_{L_2}) \sim_{[a)_{L_1}} \beta'$ and $\gamma \restriction [a)_{L_1} = \alpha$.

On the other hand,

$$h^c_{L_1,L_3}(([\gamma'']_{[a)_{L_1}},\chi''),(\alpha,\varphi)) = (\widetilde{\gamma},\widetilde{\chi}),$$

where $\widetilde{\chi} = \chi'' \cup \varphi = \chi' \cup \psi' \cup \varphi$, $\widetilde{\gamma} \sim_{[a)_{L_1}} \gamma''$, and $\widetilde{\gamma} \restriction [a)_{L_1} = \alpha$. It follows that $\widetilde{\gamma} \sim_{[b)_{L_2}} \gamma'$ and since, by the definition, $(\gamma'' \restriction [b)_{L_2}) \sim_{[a)_{L_1}} \beta'$, we have that also $(\widetilde{\gamma} \restriction [b)_{L_2}) \sim_{[a)_{L_1}} \beta'$. Thus $\widetilde{\gamma} = \gamma$ and $\widetilde{\chi} = \chi$. ∎

LEMMA 4.6: *Let L_1 be a proper $(0,1)$-sublattice of a finite distributive lattice L_2. Then $C^{a,b}_{L_1,L_2} \neq \emptyset$ iff $b \leq a$, for every $a \in J(L_1)$ and $b \in J(L_2)$.*

Proof: ($\Rightarrow$) It follows directly from the definition. ($\Leftarrow$) Suppose that $a \geq b$. Let β' be any total order on the set $[b)_{L_2}$ such that $[a)_{L_1} \in H(\beta' \restriction ([b)_{L_2} \cap L_1))$.

Define $\psi'(L_2) = [b)_{L_2}$ (it is exactly here that we use the assumption $L_1 \neq L_2$), while $\psi'(K) = \emptyset$ for every $K \subsetneq L_2$ from $\mathcal{P}(L_2) \setminus \mathcal{P}(L_1)$. It is straightforward that $([\beta']_{[a)_{L_1}}, \psi') \in C^{a,b}_{L_1,L_2}$. ∎

THEOREM 4.7: *Every distributive* $(0,1)$*-lattice is isomorphic to the semilattice of finitely generated ideals of some locally matricial algebra.*

Proof: Let $\mathcal{L}$ be a distributive $(0,1)$-lattice. Denote by P the poset of all $(0,1)$-sublattices of $\mathcal{L}$ ordered by inclusion. For all $L_1 \subseteq L_2$ in P denote by i_{L_1,L_2} the inclusion map. If the lattice $\mathcal{L}$ is finite, the assertion follows from Theorem 3.1. Suppose that $\mathcal{L}$ is infinite. Then P has no maximal elements and

$$\mathcal{L} \simeq \varinjlim \langle L_1, i_{L_1,L_2} \rangle_{L_1 \subseteq L_2 \text{ in } P}.$$

It follows from Lemma 4.5 that

$$\langle B_{L_1}, [C_{L_1,L_2}, h_{L_1,L_2}] \rangle_{L_1 \subsetneq L_2 \text{ in } P}$$

is a direct system in the category **c**. Let $L_1 \subsetneq L_2$ in P. By Lemma 4.6, $C^{a,b}_{L_1,L_2} \neq \emptyset$ iff $b \leq a$, for every $a \in J(L_1)$, and $b \in J(L_2)$. It follows that the morphism $[C_{L_1,L_2}, h_{L_1,L_2}]$ is i_{L_1,L_2}-induced with respect to identity maps. Finally, we apply Proposition 2.10. ∎

We have proved (Theorem 3.1, Theorem 3.2, Theorem 4.5) that every distributive $(\vee, 0, 1)$-semilattice which is either

(a) countable or

(b) strongly distributive or

(c) a lattice

can be represented as the semilattice of all finitely generated ideals of some unital locally matricial algebra. It is easy to observe how these results imply that every distributive $(\vee, 0)$-semilattice which is either countable or strongly distributive or a lattice is isomorphic to the semilattice of finitely generated ideals of a locally matricial algebra, now not necessarily with a unit element. Indeed, for a semilattice S, we denote by $\hat{S}$ the semilattice obtained by adding to S a new element 1 such that $1 > s$ for every $s \in S$. If S is a distributive $(\vee, 0)$-semilattice satisfying (a), (b) or (c), then $\hat{S}$ is a $(\vee, 0, 1)$-semilattice satisfying (a), (b) or (c), respectively. Then there exists a locally matricial algebra R with $\mathrm{Id}^{c}(R) \simeq \hat{S}$. The algebra R has a unique maximal two-sided ideal I which itself is a (non-unital) locally matricial algebra and the semilattice of its finitely generated two-sided ideals is isomorphic to S.

5. The Γ-invariant problem

In this section we show how to solve the Γ-invariant problem applying the main results of Section 4. The idea of the use of the Γ-invariants to classify uniform modules over associative rings is due to J. Trlifaj [T1, T2] and P. C. Eklof [ET]. We outlined the idea in the Introduction; now we are going to study it in detail.

Definition: Let L be a $(0,1)$-lattice.

(i) Let σ be a nonzero ordinal number. A sequence $\mathcal{A} = (a_\alpha \mid \alpha < \sigma)$ of nonzero elements of L is called a **cofinal strictly decreasing chain** (or c.d.c.) if
   (1) $a_{\alpha+1} < a_\alpha$ for all $\alpha < \sigma$,
   (2) $a_\beta = \bigwedge_{\alpha<\beta} a_\alpha$ for all limit ordinals $\beta < \sigma$,
   (3) if $0 \neq a \in L$, then there is $\alpha < \sigma$ such that $a_\alpha \leq a$.

(ii) The lattice L is called **strongly dense** provided L possesses a c.d.c. The **dimension** of a strongly dense lattice L is the minimal length of a c.d.c. in L.

Definition: Let L be a $(0,1)$-lattice. Let $a < b < 1$ be elements of L. Then b is **complemented over** a if there is $c \in L$ such that $b \wedge c = a$ and $b \vee c = 1$.

Definition: Let L be a strongly dense modular lattice of uncountable dimension κ. Let $\mathcal{A} = (a_\alpha \mid \alpha < \kappa)$ be a c.d.c. in L. Put

$$E(\mathcal{A}) = \{\alpha < \kappa \mid \exists_{\beta>\alpha} \colon a_\alpha \text{ is not complemented over } a_\beta\}.$$

Denote by $B(\kappa)$ the Boolean algebra of all subsets of κ modulo the filter generated by closed unbounded sets. Given a subset E of κ, we denote by $\overline{E}$ the element of $B(\kappa)$ represented by E. The equivalence class $\overline{E(\mathcal{A})}$ does not depend on a particular choice of a c.d.c. of the minimal length κ [ET, Lemma 1.8]. It is called the *Γ*-**invariant**, $\Gamma(L)$, of the strongly dense lattice L.

Let κ be a regular uncountable cardinal and let E be a subset of $\kappa \setminus \{\emptyset\}$. Let L_E be the lattice defined in [ET, Definition 1.12], that is, the $(0,1)$-sublattice of the lattice of all subsets of κ ordered by inverse inclusion generated by intervals $[\alpha, \beta)$, where $\alpha < \beta < \kappa$ and $\alpha \notin E$. By [ET, Theorem 1.13], L_E is a strongly dense distributive lattice of cardinality and dimension κ such that $\Gamma(L_E) = \overline{E}$. Denote by I_E the ideal lattice of L_E. By [ET, Theorem 1.15], I_E is a strongly dense algebraic distributive lattice of dimension κ whose greatest element is compact and $\Gamma(I_E) = \overline{E}$.

Let L be a modular lattice. Then

$$\{a \in L \mid b \text{ is not complemented over } a\}$$

is a lower subset of L for every nonzero element $b \in L$ [ET, Lemma 1.4]. A nonzero element b of the lattice L is called **weakly complemented** if b is complemented over a for every a with $0 < a < b$.

Definition: Let L be a strongly dense lattice of dimension $\kappa > 1$.

(i) L is **complementing** provided L possesses a c.d.c. $\mathcal{A} = (a_\alpha \mid \alpha < \kappa)$ such that for all $\alpha < \beta < \kappa$, a_α is complemented over a_β.

(ii) L is **narrow** provided that it is not complementing and L possesses a c.d.c. $\mathcal{A} = (a_\alpha \mid \alpha < \kappa)$ such that for all $\alpha < \beta < \kappa$, a_α is not complemented over a_β.

(iii) L is **constricted** provided that it does not have a c.d.c. $\mathcal{A} = (a_\alpha \mid \alpha < \kappa)$ such that for all $\alpha < \kappa$, $a_{\alpha+1}$ is weakly complemented.

By [ET, Theorem 1.10], a strongly dense modular lattice L of dimension κ is complementing if and only if $\Gamma(L) = \overline{\emptyset}$ and it is narrow if and only if $\Gamma(L) = \overline{\kappa}$. Due to [ET, Corollary 1.11], the lattice L is constricted if and only if there exists $a > 0$ in L such that a' is not weakly complemented for every a' with $0 < a' < a$. It follows that L is narrow provided L is constricted. On the other hand, given an uncountable regular cardinal κ, the lattice L_{E_2} where $E_2 = \{\alpha < \kappa \mid \alpha \text{ is a limit ordinal}\}$ is a narrow but not constricted distributive lattice of dimension κ [ET, Corollary 1.14].

An R-module M is called **strongly uniform** provided the lattice $L(M)$ of its submodules is strongly dense. The dimension and the Γ-invariant of a strongly uniform module M correspond to the dimension and the Γ-invariant of the lattice $L(M)$. A strongly uniform module M is **complementing, narrow**, or **constricted** if the lattice $L(M)$ is complementing, narrow, or constricted. The following problems are stated in [ET]:

[ET, Problem 2.3]: *For an uncountable regular cardinal κ, which elements of $\mathcal{B}(\kappa)$, other than $\overline{\kappa}$, are the Γ-invariant of a strongly uniform module over a regular ring?*

[ET, Problem 2.4]: *Is there a strongly uniform module of dimension κ which is narrow but not constricted?*

Both the problems are solved combining Theorem 4.5 and [ET, Lemma 2.1]:

[ET, LEMMA 2.1]: *Let L be an algebraic lattice and k be a field. Assume that $L \simeq \mathrm{Id}(S)$ for a k-algebra S. Then $L \simeq L(M)$ for some right R-module M, where $R = S \otimes_k S^{op}$. Moreover, if S is a locally matricial k-algebra, then so is R.*

THEOREM 5.1: *Let κ be an uncountable regular cardinal, let E be a subset of $\kappa \setminus \{0\}$. Then there exists a locally matricial algebra R and a right R-module M with $L(M) \simeq I_E$.*

In particular, all elements of $\mathcal{B}(\kappa)$ are realized as the Γ-invariant of a strongly uniform module over a unit-regular ring.

Proof: Since ${I_E}^{\mathrm{c}} \simeq L_E$, compact elements of I_E form a distributive lattice. By Theorem 4.5, there exists a locally matricial algebra S with $\mathrm{Id}^{\mathrm{c}}(S) \simeq L_E$, whence $\mathrm{Id}(S) \simeq I_E$. Now, by [ET, Lemma 2.1], $L(M) \simeq I_E$ for a right $R = S \otimes S^{op}$-module M, and R is a locally matricial algebra. ∎

THEOREM 5.2: *For every uncountable regular cardinal κ there exists a strongly uniform module of dimension κ, over a locally matricial algebra, which is narrow but not constricted.*

Proof: Let

$$E_2 = \{\alpha < \kappa \mid \alpha \text{ is a limit ordinal}\}.$$

Then the algebraic lattice I_{E_2} is narrow but not constricted. By Theorem 5.1, there are a locally matricial algebra R and a right R-module M with $L(M) \simeq I_{E_2}$. ∎

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